Entropy as the basic rule of natural and sociological science - Part (I)

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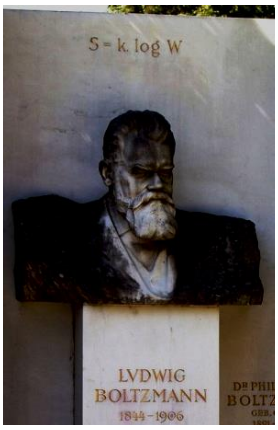
On the grave of Ludwig Boltzmann in Vienna Central Cemetery one can see the famous inscription of an equation as shown below which reads,

"S=klogW "



This is one of the monumental works of Boltzmann telling us what is entropy in statistical thermodynamics. This formula tells that entropy, S, is logarithm of the number of states one atom can take (density of states), W, multiplied by a constant k the Boltzmann constant. Since k is a constant if S is normalized by k the equation may simply be expressible as, S = logW, without changing the meaning of the formula.

Although this equation is generally understood as the formula for thermodynamics, since it is a mathematical equation, S may be interpretable as the utility of any number N that belongs to certain object, person or subject one is concerned.

For instance, if N is the number of apples per person, the utility of that thing can be written as,

S = logN

To understand the meaning of this expression, we must remind ourselves the meaning of logarithm of a number. A short reminder of logarithmic counting method is given below.

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We count numbers one, two, three, four... and so on. If you think ten is much greater than one, one thousand is much much much greater than one. OK then how about one million? It must be much, much, much... total of 6 muches greater than one. So, there is a way of counting numbers instead of number itself but by counting how many "much"es do you feel by these numbers in comparison with 1 (= zero much state).

Mathematical expression of the counting numbers as explained above can be written as;

$10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$......$10^6 = 1000000$...and so on.

In short there is a way of counting numbers not by the number itself but by counting the exponents of numbers or in other words by counting the number which indicates the order of magnitude of the number.

The way to count number N by the order of magnitude instead of the conventional way is called the logarithm of the number and denoted as logN.

Hence, log1 =0, log10 = 1, log100 = 2, log1000 = 3, ... log1000000 = 6 and so on.

Mathematically, any arbitrary number such as 2, 5, 10, 20 ... can also be counted by the logarithmic way as,

log2 = 0.301.., log 5 = 0.698.., log10 = 1, log20 = 1.30....

Please note that the logarithmic number increases much slower than the usual number (you have seen that 1000000, one million, is only 6 when counted by logarithm).

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When one accepts the definition of entropy following the Boltzmann's equation S = logN, one must understand at least in principle how this relation is resulted from the rule of nature.

For this purpose, we should see how much new utility arises when the number N is increased by 1. Such question can quite easily be answered mathematically by the differentiation of that equation. Well, differentiation in short is nothing but the way to find out how much change the given function (S = logN) suffers when N is increased by 1. Any high-school student can see it (d(logN)/dN = 1/N) but the way how is not the problem here. The answer is the increase of S when N is increased by 1 is proportional to 1/N.

Now, the reason why the utility of number N which is S increases in proportion to 1/N due to the increase of N by 1 is readily understandable since the ratio of number 1 to whole N decreases in proportion to 1/N. One may understand that it is the ratio of 1 to whole N which determines the impact of addition of 1 to N.

It may be interesting to many people to see the following examples.

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The price of one apple P is inversely proportional to the number of apples per person N available in some locality. Thus, merchants purchase these in the country where apple per person is N_2 at P_2 which equals to $1/N_2$ and sell them in town, where there are N_1 available per person, at P_1 which equals to $1/N_1$. The profit ratio, η, is calculable by dividing the difference between the selling price and the purchasing price by the selling price as,

$\eta = (P_1 - P_2)/ P_1 = (1/ N_1 - 1/ N_2)/ (1/ N_1) = (1 - N_1/ N_2)$

Thus, the profit ratio depends on the ratio in availability of apples per person in the country and in town. This expression is exactly same as the Carnot efficiency, $\eta = (1 - T_1/ T_2)$, expressing the proportion of energy available to be used for driving thermal engines by transferring heat energy from the high temperature T_2 to the low temperature T_1. One can understand that how high is the temperature T, is the same notion as how numerous are the apples (heat energy) per person (one atom).

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It is rather surprising to see these two phenomena, one is dependent- on human sense of the utility of apples and the other is strictly mechanical natural law of utility of heat energy, are expressible (or governed) by the same rule of mathematics which leads to the notion of entropy.

It may interest some people if you fancy you can count happiness by numbers. If N is the total number of one's happiness, the increase of one's happiness due to the addition 1 happiness is not 1 but is 1/N. One can readily imagine that the larger the N the smaller the impact you gain by the addition of 1 happiness to oneself. Everybody would understand that when one who is scares in happiness (small N) the addition of only 1 happiness or even 0.1 should give one the great feeling of happiness.

Many of the reader of this article may already have noticed that the "law of decreasing marginal utility" in economics tells exactly same notion described above for apple price, happy feeling and entropy in thermodynamics. Exactly same equation as Boltzmann's entropy was already written in Daniel Bernoulli's paper in 1738 which treated the magnitude of feeling of utility and the actual money earned by betting, commonly called as St. Petersburg problem.

Thus, the notion of entropy inscribed on Boltzmann's grave is not restricted to thermodynamics but must be understood as a ubiquitous natural rule which governs both scientific and sociological phenomena.